Limit of Functions

1. Prove the followings:

(a) $\lim_{x\to\pi}\frac{\sin mx}{\sin nx}=\frac{m}{n}(-1)^{m-n}, m,n\in\mathbf{N}$

(b) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

(c) $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}=\frac{2}{\pi}$

(d) $\lim_{x\to 0}\frac{\cos x-\cos 3x}{x^2}=4$

(e) $\lim_{x\to\frac{\pi}{6}}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}=-3$

(f) $\lim_{x\to\frac{\pi}{3}}\frac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x}=\frac{1}{\sqrt{3}}$

2. Prove the following:

(a) $\lim_{x\to 0}\frac{x^2-1}{2x^2-x-1}=1$

(b) $\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}=\frac{2}{3}$

(c) $\lim_{x\to\infty}\frac{x^2-1}{2x^2-x-1}=\frac{1}{2}$

3. Prove the following:

(a) $\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}=6$

(b) $\lim_{x\to 0}\frac{(1+x)^5-(1+5x)}{x^2+x^5}=10$

(c) $\lim_{x\to 1}\frac{x+x^2+....+x^n-n}{x-1}=\frac{n(n+1)}{2}$

(d) $\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\frac{m-n}{2}$, m,n$\in$**N**

4. Prove the following:

(a) $\lim_{x\to 1^-}\frac{1}{1+e^{\frac{1}{x}}}=1$

(b) $\lim_{x\to 1^+}\frac{1}{1+e^{\frac{1}{x}}}=0$

(c) $\lim_{x\to -\infty}\frac{\ln(1+e^x)}{x}=0$

(d) $\lim_{x\to +\infty}\frac{\ln(1+e^x)}{x}=1$

5. Prove the following:

(a) $\lim_{x\to+\infty}\left[\sqrt{(x+a)(x+b)}-x\right]=\frac{1}{2}(a+b)$

(b) $\lim_{x\to+\infty}\left(\sqrt[3]{x^3+x^2+1}-\sqrt[3]{x^3-x^2+1}\right)=\frac{2}{3}$

(c) $\lim_{x\to\infty}x^{\frac{1}{3}}\left[(x+1)^{2/3}-(x-1)^{2/3}\right]=\frac{4}{3}$

(d) $\lim_{x\to 64}\frac{\sqrt[3]{x}-4}{\sqrt{x}-8}=\frac{1}{3}$

(e) $\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}=\frac{12}{5}$

6. Let $\lim_{x\to a}f(x)=L$, prove that $\lim_{x\to a}|f(x)|=|L|$.

Also prove that, unless $L=0$, the converse does not hold.

7 Show that the function $f(x)=\frac{x\tan x+2x-1}{x+1}$ does not have a unique limit as $x\to\infty$ through all real values, but $f(x)\to 3$ as x ranges through the sequence $\left\{n\pi+\frac{\pi}{4}\right\}$, n = 0, 1, 2, ….

8. Show that the function $\phi(x)=\lim_{n\to\infty}\frac{f(x)+ng(x)\sin^2\pi x}{1+n\sin^2\pi x}$ is equal to $f(x)$ where x is an integer, but is equal to $g(x)$ in every other cases.

9. Let f(x), g(x) be two continuous functions defined on **R**, and: $\phi(x)=\begin{cases} f(x) & \text{where } x^2>1 \\ g(x) & \text{where } x^2<1 \\ \frac{1}{2}[f(x)+g(x)] & \text{where } x=1 \end{cases}$

Show that $\phi(x)=\lim_{n\to\infty}\frac{x^n f(x)+g(x)}{x^n+1}$, $x \neq -1$. Is $\phi(x)$ continuous on **R** ?

10. Show that if $\lim_{x\to\infty} f(x)=0$ and $f(x)\neq 0$, then $\lim_{x\to\infty}\left|\frac{1}{f(x)}\right|=\infty$.

11. Show that if $\lim_{x\to\infty} f(x)=\infty$, then $\lim_{x\to\infty}[f(x)]^2=\infty$.

12. Is it true that if $\lim_{x\to\infty} f(x)=a$, then $\lim_{x\to-\infty} f(x)=-a$.

13. If $\lim_{x\to a} f(x)=\pm\infty$, can we have $\lim_{x\to a} f(x)=-\infty$.

14. Can we have $\lim_{x\to a} f(x)=m$ and $\lim_{x\to a} f(x)=n$ $(m\neq n)$ simultaneously?

15. Can we have $\lim_{x\to a} f(x)=f(a)$ all the time?

16. If f is defined by $f(x)=\begin{cases} x & \text{if } x\neq 1 \\ 3 & \text{if } x=1 \end{cases}$, sketch the graph of f and find $\lim_{x\to 1} f(x)$.

17. If $f(x)=\begin{cases} 2 & \text{if } x\neq 2 \\ 3 & \text{if } x=2 \end{cases}$, sketch the graph of f and find $\lim_{x\to 2} f(x)$.

18. sgn(x) is defined by $\operatorname{sgn}(x)=\begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$, sketch the graph of sgn(x) and find $\lim_{x\to a} \operatorname{sgn}(x)$.

19. Graph f(x) = [x], [x] is the greatest integer less than x. Find $\lim_{x\to a} f(x)$.

20. **(a)** Prove by induction that for $n \geq 10$, $2^n > n^3$.

(b) Find $\lim_{x\to\infty}\frac{2^x}{3x^2}$ by considering n the greatest integer smaller than x .

21. Find the following limits:

(a) $\lim_{x\to 0}\frac{x^2-1}{2x^2-x-1}$

(b) $\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}$

(c) $\lim_{t\to 1}\frac{t^2-t^{\frac{1}{2}}}{t^{\frac{1}{2}}-1}$

(d) $\lim_{x\to 0}\frac{(1+mx)^n-(1+nx)^n}{x^2}$

(e) $\lim_{x\to 0}\frac{\sin mx}{\sin nx}$

(f) $\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$

(h) $\lim_{x\to-\infty}\left(\sqrt{x^2+1}-x\right)$

(i) $\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$

(j) $\lim_{x\to0}\frac{1+\sin x-\cos x}{1+\sin px-\cos px}$

(k) $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}$, where p, q are polynomials

22. Give an example of f with $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)\neq f(a)$.

23. Give an example of f and g so that $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=\infty$, but $\lim_{x\to a}[f(x)-g(x)]$ does not exist.

24. Give an example of f and g so that $\lim_{x\to a} f(x)=0$, $\lim_{x\to a} g(x)=\infty$, but $\lim_{x\to a}\frac{g(x)}{f(x)}$ does not exist.

25. Give an example of f so that $\lim_{x\to a}|f(x)|$ exists but not for $\lim_{x\to a} f(x)$.

26. Find the left and right limits of

(a) $f(x)=\begin{cases}0 & ,x>1\\ 1 & ,x=1\\ x^2+2x & ,x<1\end{cases}$ $(x=1)$

(b) $f(x)=\begin{cases}0 & x\in\mathbf{Q}\\ 1 & x\notin\mathbf{Q}\end{cases}$ $(x=\frac{1}{2})$

27. If it is known that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$,

(a) Show that $\lim_{n\to\infty}\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right)=e$

(b) Hence show that $e=1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\frac{\theta_n}{n!n}$, where $0<\theta_n<1$.

(c) Calculate e to 5 decimal places.

(d) Evaluate :

(i) $\lim_{n\to\infty}\left(1-\frac{2}{3n}\right)^n$ (ii) $\lim_{x\to\infty}\left(\frac{x^2+a}{x^2-a}\right)^{x^2}$ (iii) $\lim_{n\to\infty}\left(\frac{n}{a+n}\right)^n$ (iv) $\lim_{x\to0}(1-x)^{1/x}$

(e) Evaluate $\lim_{n\to\infty}\left(1+\frac{1}{P_n}\right)^{P_n}$, where $\lim_{n\to\infty}P_n=\infty$

(f) Evaluate: $\lim_{n\to\infty}\left(\frac{an+b}{an+c}\right)^n=e$

(g) Evaluate $\lim_{x\to0}(1+x)^{\frac{b}{x}}$, and hence find $\lim_{x\to\infty}\left(\frac{1-\frac{2}{x}-\frac{3}{x^2}}{1-\frac{3}{x}-\frac{28}{x^2}}\right)^x$